

Thurston's bending measure conjecture for once punctured torus groups

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Abstract

We prove Thurston's bending measure conjecture for quasifuchsian once punctured torus groups. The conjecture states that the bending measures of the two components of the convex hull boundary uniquely determine the group.

1 Introduction

Thurston conjectured that a convex hyperbolic structure on a 3-manifold with boundary is uniquely determined by the bending measure on the boundary. In this paper we prove the conjecture in one of the simplest possible examples, namely when the manifold is an interval bundle over a once punctured torus.

In [12], the author and L. Keen studied the convex hull boundary of this class of hyperbolic 3-manifolds in great detail, without however addressing Thurston's conjecture *per se*. Since the question came up several times during the Newton Institute programme, it seemed worthwhile to investigate, even though the once punctured torus is a very special case. Given the current state of knowledge, the basic idea of the present proof is rather simple. It does however build on a large number of rather deep results from [12] and elsewhere.

Let $G \subset PSL(2, \mathbb{C})$ be quasifuchsian, so that the corresponding quotient 3-manifold \mathbb{H}^3/G is an interval bundle over a surface S . The boundary of the convex core of \mathbb{H}^3/G has two components $\partial\mathcal{C}^\pm$ each of which are pleated surfaces bent along measured laminations β^\pm on S called the *bending measures*. The claim of the bending measure conjecture is that β^\pm uniquely determine G , up to conjugation in $PSL(2, \mathbb{C})$. If the underlying supports of β^\pm are closed curves, then a result of Bonahon and Otal [3] (which applies to general Kleinian groups not just the quasifuchsian case under discussion here) asserts this is indeed the case. They also gave necessary and sufficient conditions for existence, but not uniqueness, of quasifuchsian groups for which the bending measures β^\pm are any given pair of bending

measures μ, ν . If S is a once punctured torus, these conditions reduce to $i(\mu, \nu) > 0$ and the proviso that the weight of any closed curve is less than π . In a recent preprint [1], Bonahon has also shown uniqueness for general quasifuchsian groups which are sufficiently close to being Fuchsian.

The object of [12] was to describe the space \mathcal{QF} of quasifuchsian groups when the surface S is a once punctured torus in terms of the geometry of $\partial\mathcal{C}^\pm$. This was done by analysing the *pleating planes* $\mathcal{P}(\mu, \nu)$ consisting of all groups whose bending laminations lie in a particular pair of projective classes $[\mu], [\nu]$, see Section 2. In place of the bending measures, we concentrated on the lengths l_μ and l_ν of μ and ν , which can be extended to well-defined holomorphic functions, the *complex lengths* l_μ and l_ν , on \mathcal{QF} , see Section 3. We showed, see Theorem 3.1, that l_μ and l_ν are local holomorphic coordinates for \mathcal{QF} in a neighbourhood of $\mathcal{P}(\mu, \nu)$ whose restrictions to $\mathcal{P}(\mu, \nu)$ are real valued. In this way we obtained a diffeomorphism of $\mathcal{P}(\mu, \nu)$ with a certain open subset in $\mathbb{R}^+ \times \mathbb{R}^+$. In particular, the lengths l_μ and l_ν uniquely determine the group.

Given this background, to prove the bending measure conjecture we just need to show that, restricted to a given pleating variety, the map from lengths to bending measures is injective. (Here by bending measure, we really mean the scale factors which relate β^\pm to a fixed choice of laminations $\mu \in [\mu], \nu \in [\nu]$.) For rational laminations, as mentioned above, we already know by [3] that angles determine the group. In [6], again in the context of general Kleinian groups, we showed that for rational bending laminations the map from bending angles to lengths is injective and moreover that its Jacobian is symmetric and negative definite. If S is a once punctured torus, it follows (see Section 5) that if both bending laminations are rational and if we fix the length of the bending line on say $\partial\mathcal{C}^+$, then the bending angle on $\partial\mathcal{C}^-$ is a monotonic function of the length on $\partial\mathcal{C}^-$.

Now keeping the bending line on $\partial\mathcal{C}^+$ fixed and of fixed length, we take limits as the bending laminations on $\partial\mathcal{C}^-$ converge projectively to an arbitrary irrational lamination ν . Using the fact that the limit of monotone functions is monotone, we deduce the scale factor of the bending measure on ν is still a monotone function of the length l_ν . Since this scale factor is real analytic, it is either strictly monotonic or constant; a global geometrical argument rules out constancy. An elaboration of the argument in Sections 6 and 7 then allows us relax the requirement that the bending lamination and length on $\partial\mathcal{C}^+$ remain fixed and prove the conjecture.

In [12] we also showed that the rational pleating varieties for which the supports of the bending laminations are simple closed curves are dense. This is not enough for the present proof: to compare our monotone functions properly, we need the “limit pleating theorem” of [12], see Theorem 3.3. This is a deep result closely related to the “lemme de fermeture” in [3]. Roughly, it asserts the existence of an algebraic limit in $\mathcal{P}(\mu, \nu)$ for any sequence along which the pleating lengths l_μ, l_ν remain bounded.

A simpler version of the same analysis, not spelled out here, would show that the bending angle also uniquely determines the group for the so-called Riley slice of Schottky

space, see [13].

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2 Preliminaries

Throughout the paper, S will denote a fixed topological once punctured torus. A representation $\pi_1(S) \rightarrow PSL(2, \mathbb{C})$ is called quasifuchsian if the image is discrete and torsion free, if the image of a simple loop around the puncture is parabolic, and if the quotient hyperbolic manifold $M = \mathbb{H}^3/G$ is homeomorphic to $S \times (-1, 1)$. It is Fuchsian if it is quasifuchsian and if in addition the representation is conjugate to a representation into $PSL(2, \mathbb{R})$. Let \mathcal{F} and \mathcal{QF} denote the spaces of Fuchsian and quasifuchsian representations respectively, modulo conjugation in $PSL(2, \mathbb{C})$. The space \mathcal{QF} is a smooth complex manifold of dimension 2, with natural holomorphic structure induced from $PSL(2, \mathbb{C})$.

2.0.1 Measured laminations

We assume the reader is familiar with geodesic laminations, see for example [22, 19]. A measured lamination μ on S consists of a geodesic lamination, called the support of μ and denoted $|\mu|$, together with a transverse measure, also denoted μ . We topologise the space \mathcal{ML} of all measured laminations on S with the topology of weak convergence of transverse measures, that is, two laminations are close if the measures they assign to any finite set of transversals are close. We write $[\mu]$ for the projective class of $\mu \in \mathcal{ML}(S)$ and denote the set of projective equivalence classes on S by \mathcal{PML} .

Let \mathcal{S} be the set of simple closed curves on S . We call $\mu \in \mathcal{ML}$ *rational* if $|\mu| \in \mathcal{S}$. (On a more general surface, we say a lamination is rational if its support is a disjoint union of simple closed curves.) Equivalently, μ is rational if $\mu = c\delta_\gamma$ where $\gamma \in \mathcal{S}$, $c \geq 0$ and δ_γ is the measured lamination which assigns unit mass to each intersection with γ . The set of all rational measured laminations on S is denoted $\mathcal{ML}_\mathbb{Q}$; this set is dense in \mathcal{ML} .

The geometric intersection number $i(\gamma, \gamma')$ of two geodesics $\gamma, \gamma' \in \mathcal{S}$ extends to a jointly continuous function $i(\mu, \nu)$ on \mathcal{ML} . It is special to the once punctured torus that $i(\mu, \nu) > 0$ is equivalent to $[\mu] \neq [\nu]$, moreover (since all laminations on S are uniquely ergodic) if $\mu, \nu \in \mathcal{ML}$ with $|\mu| = |\nu|$, then $[\mu] = [\nu]$.

For general surfaces, convergence of laminations in the topology of \mathcal{ML} does not imply Hausdorff convergence of their supports. However:

Lemma 2.1 ([12] Lemma 1) *Suppose S is a once punctured torus. Suppose that $\mu \notin \mathcal{ML}_\mathbb{Q}$ and that $\mu_n \rightarrow \mu$ in the topology of weak convergence on \mathcal{ML} . Then $|\mu_n| \rightarrow |\mu|$ in the Hausdorff topology on the set of closed subsets of S .*

2.0.2 Lengths

Given a hyperbolic structure on S (associated to a Fuchsian representation of $\pi_1(S)$) and $\gamma \in \pi_1(S)$, we define the length l_γ to be the hyperbolic length of the unique geodesic freely homotopic to γ . This definition can be extended to general laminations $\mu \in \mathcal{ML}$. The following theorem summarizes the results of [14], Lemma 2.4 and [16], Theorem 1:

Proposition 2.2 *The function $(c\delta_\gamma, p) \mapsto cl_{\delta_\gamma}(p)$ from $\mathcal{ML}_\mathbb{Q} \times \mathcal{F}$ to \mathbb{R}^+ extends to a real analytic function $(\mu, p) \mapsto l_\mu(p)$ from $\mathcal{ML} \times \mathcal{F}$ to \mathbb{R}^+ . If $\mu_n \in \mathcal{ML}_\mathbb{Q}$, $\mu_n \rightarrow \mu$ then $l_{\mu_n}(p) \rightarrow l_\mu(p)$ uniformly on compact subsets of \mathcal{F} .*

We showed in [12] that for $\mu \in \mathcal{ML}$, the length function l_μ on \mathcal{F} extends to a non-constant holomorphic function λ_μ , called the *complex length* of μ , on \mathcal{QF} . The extension is done in such a way that $\lambda_{c\mu} = c\lambda_\mu$ for $c > 0$, and such that if $\mu = \delta_\gamma$ then $q \mapsto \lambda_\mu(q)$ is a well-defined branch of the complex length $2\cosh^{-1} \text{Tr } \rho(\gamma)/2$, where $\rho : \pi_1(S) \rightarrow \text{PLS}(2, \mathbb{C})$ represents $q \in \mathcal{QF}$.

Proposition 2.3 ([12] Theorem 20) *The family $\{\lambda_\mu\}$ is uniformly bounded and equicontinuous on compact subsets of $\mathcal{ML} \times \mathcal{QF}$, in particular if $\mu_n \rightarrow \mu \in \mathcal{ML}$ and $q_n \rightarrow q \in \mathcal{QF}$ then $\lambda_{\mu_n}(q_n) \rightarrow \lambda_\mu(q)$.*

The real part l_μ of the complex length l_μ is the lamination length of μ in \mathbb{H}^3/G as defined in [22] p. 9.21.

2.0.3 The Thurston boundary

We recall the fundamental inequality which governs Thurston's compactification of Teichmüller space \mathcal{F} with projective measured lamination space \mathcal{PML} , see [9] Lemme II.1 and [23] Theorem 2.2:

Proposition 2.4 *Suppose $\sigma_n \rightarrow [\xi] \in \mathcal{PML}$. Then there exist $C > 0$, $d_n \rightarrow \infty$ and $\xi_n \rightarrow \xi$ in \mathcal{ML} such that*

$$d_n i(\xi_n, \zeta) \leq l_\zeta(\sigma_n) \leq d_n i(\xi_n, \zeta) + Cl_\zeta(\sigma_0)$$

for all $\zeta \in \mathcal{ML}$.

Corollary 2.5 *Suppose given a sequence of measured laminations μ_n on the once punctured torus S such that $\mu_n \rightarrow \mu$ in \mathcal{ML} , and suppose that $\sigma_n \in \mathcal{F}$ is a sequence of metrics such that $l_{\mu_n}(\sigma_n) \rightarrow 0$. Then $\sigma_n \rightarrow [\mu] \in \mathcal{PML}$.*

PROOF. If σ_n converged to a point σ_∞ in \mathcal{F} then $l_\mu(\sigma_\infty) = \lim_n l_{\mu_n}(\sigma_n) = 0$ which is impossible. Thus σ_n converges to some point $[\xi] \in \mathcal{PML}$. Substituting in the fundamental inequality we obtain $\xi_n \rightarrow \xi$ in \mathcal{ML} and $d_n \rightarrow \infty$ such that:

$$d_n i(\xi_n, \mu_n) \leq l_{\mu_n}(\sigma_n) \leq d_n i(\xi_n, \mu_n) + Cl_{\mu_n}(\sigma_0).$$

Since $i(\xi_n, \mu_n) \rightarrow i(\xi, \mu)$, we see that $l_{\mu_n}(\sigma_n)/d_n \rightarrow i(\xi, \mu)$. We conclude that $i(\xi, \mu) = 0$, and hence, since S is a punctured torus, that $[\xi] = [\mu]$. \square

2.0.4 Bending measures

Let $q \in \mathcal{QF}$ and let $G = G(q)$ be a group representing q . The convex core \mathcal{C} of \mathbb{H}^3/G is the smallest closed set containing all closed geodesics; it is the projection to the quotient of the convex hull of the limit set and has non-zero volume if and only if $G \in \mathcal{QF} - \mathcal{F}$. In this case its boundary has two connected components $\partial\mathcal{C}^\pm$ each of which is homeomorphic to S . The metric induced on $\partial\mathcal{C}^\pm$ from \mathbb{H}^3/G makes each a pleated surface. The bending laminations of these surfaces carry natural transverse measures, the *bending measures* β^\pm , see [8, 10]. The underlying geodesic laminations $|\beta^\pm|$ are called the *pleating loci* of G . Since the same geodesic lamination cannot be the pleating locus on both sides, we have $i(\beta^+, \beta^-) > 0$, see [12, 3]. If β^+ is rational with support $\gamma \in \mathcal{S}$, then $\beta^+ = \theta_\gamma \delta_\gamma$ where θ_γ is the bending angle along γ . In this case we have the obvious constraint $\theta_\gamma < \pi$. If G is Fuchsian we define $\beta^\pm = 0$. One of the main results of [10] is that β^\pm are continuous functions on \mathcal{QF} .

The following central existence result is a special case of Bonahon and Otal’s “Lemme de fermeture”:

Theorem 2.6 ([3] Proposition 8) *Suppose $\mu, \nu \in \mathcal{ML}$ with $i(\mu, \nu) > 0$. If $\mu = c\delta_\gamma$ is rational assume also that $c < \pi$, and similarly for ν . Then there exists a quasifuchsian group $G \in \mathcal{QF}$ such that $\beta^+ = \mu$ and $\beta^- = \nu$. If μ, ν are rational, then G is unique.*

The object of this paper is to prove the uniqueness part of this statement for arbitrary μ and ν .

3 Pleating varieties

In this section we review the results we shall need from [12].

Fix $\mu, \nu \in \mathcal{ML}$. The (μ, ν) -pleating variety is the set $\mathcal{P}(\mu, \nu) \subset \mathcal{QF}$ such that $\beta^+ \in [\mu]$ and $\beta^- \in [\nu]$ with $\beta^\pm \neq 0$. (The last condition is equivalent to $\mathcal{P}(\mu, \nu) \cap \mathcal{F} = \emptyset$.) Notice that $\mathcal{P}(\mu, \nu)$ actually only depends on the projective classes $[\mu]$ and $[\nu]$. By Theorem 2.6 or alternatively [12] Theorem 2, $\mathcal{P}(\mu, \nu)$ is non-empty if and only if $i(\mu, \nu) > 0$. By [12] Proposition 22, the complex length λ_μ is real valued whenever the projective class of β^+ (or β^-) is $[\mu]$. In this case, $l_\mu = \Re l_\mu$ is both the lamination length in \mathbb{H}^3/G and the length of μ in the hyperbolic structure on $\partial\mathcal{C}^+$ (or $\partial\mathcal{C}^-$).

Let (μ, ν) be measured laminations on S with $i(\mu, \nu) > 0$. Then by [15], the function $l_\mu + l_\nu$ has a unique minimum on \mathcal{F} . Moreover as c varies in $(0, \infty)$, the minima of $l_\mu + l_{c\nu}$ form a line $\mathcal{L}(\mu, \nu)$; it is easy to see that this line only depends on the projective classes $[\mu]$ and $[\nu]$. The lines $\mathcal{L}(\mu, \nu)$ vary continuously with μ and ν , see [15]. (In [12] we gave a slightly different description of $\mathcal{L}(\mu, \nu)$ in terms of the minimum of l_μ on an earthquake path

along ν , namely, $\mathcal{L}(\mu, \nu)$ is the locus where $\frac{\partial l_\mu}{\partial t_\nu}$ vanishes, where t_ν denotes the earthquake along ν . This definition only works for the once punctured torus; for the equivalence of the two definitions, see [12] Lemma 6.) We proved in [21] Theorem 1.7 (in the context of arbitrary hyperbolisable surfaces S) that the closure of $\mathcal{P}(\mu, \nu)$ meets \mathcal{F} exactly in $\mathcal{L}(\mu, \nu)$.

For each $c > 0$, the line $\mathcal{L}(\mu, \nu)$ meets the horocycle $l_\mu = c$ in \mathcal{F} in exactly one point. Let $f_{\mu, \nu} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the function $f_{\mu, \nu}(c) = l_\nu(d)$ where $d = \mathcal{L}(\mu, \nu) \cap l_\mu^{-1}(c)$. We showed ([12] Lemma 6) that $f_{\mu, \nu}$ is surjective and strictly monotone decreasing. We denote by $\mathcal{R}(\mu, \nu)$ the open region in $\mathbb{R}^+ \times \mathbb{R}^+$ bounded between the two coordinate axes and the graph of $f_{\mu, \nu}$. The following description of $\mathcal{P}(\mu, \nu)$ is one of the main results of [12]:

Theorem 3.1 ([12] Theorem 2) *Let $\mu, \nu \in \mathcal{ML}$ be measured laminations with $i(\mu, \nu) > 0$. Then the function $l_\mu \times l_\nu$ is locally injective in a neighbourhood of $\mathcal{P}(\mu, \nu)$, moreover its restriction to $\mathcal{P}(\mu, \nu)$ is a diffeomorphism to $\mathcal{R}(\mu, \nu)$.*

This means that $\mathcal{P}(\mu, \nu)$ is totally real, in other words, there are local holomorphic coordinates such that a neighbourhood of $x \in \mathcal{P}(\mu, \nu) \hookrightarrow \mathcal{QF}$ is identified with a neighbourhood of $0 \in \mathbb{R}^2 \hookrightarrow \mathbb{C}^2$. Moreover $\mathcal{P}(\mu, \nu)$ is a connected real 2-manifold, on which we can take (l_μ, l_ν) as global real analytic coordinates, where as above we define $l_\mu = \Re l_\mu, l_\nu = \Re l_\nu$.

Corollary 3.2 *The set $L_c = l_\nu^{-1}(c) \subset \mathcal{P}(\mu, \nu)$ is connected and hence can be regarded as a line in $\mathcal{R}(\mu, \nu)$ parameterised by l_μ .*

PROOF. This is immediate from the fact that $f_{\mu, \nu}$ is strictly monotonic. \square

A crucial ingredient of Theorem 3.1 was the following, which we call the *limit pleating theorem*:

Theorem 3.3 ([12] Theorem 5.1) *Suppose $q_n \in \mathcal{P}(\mu, \nu)$ is a sequence such that the lengths $l_\mu(q_n), l_\nu(q_n)$ are uniformly bounded above. Then there is a subsequence of q_n such that the corresponding groups $G(q_n)$ converge algebraically to a group G_∞ . If both $\lim_n l_\mu(q_n)$ and $\lim_n l_\nu(q_n)$ are strictly positive, then $G_\infty \in \mathcal{F} \cup \mathcal{P}(\mu, \nu)$ and the convergence is strong.*

It is convenient to have a somewhat stronger version of this theorem, in which we allow the bending loci to vary also:

Theorem 3.4 *Suppose that $\mu, \nu \notin \mathcal{ML}_\mathbb{Q}$ and that $\mu_n \rightarrow \mu$ and $\nu_n \rightarrow \nu$. Suppose $q_n \in \mathcal{P}(\mu_n, \nu_n)$ and that l_{μ_n}, l_{ν_n} are uniformly bounded above. Then there is a subsequence of q_n such that $G(q_n)$ converges algebraically to a group G_∞ . If both $\lim_n l_{\mu_n}(q_n)$ and $\lim_n l_{\nu_n}(q_n)$ are strictly positive, then $G_\infty \in \mathcal{F} \cup \mathcal{P}(\mu, \nu)$ and the convergence is strong.*

The proof is almost identical to that in [12]. We make the assumption that $\mu, \nu \notin \mathcal{ML}_\mathbb{Q}$ since we need that the convergence of μ_n to μ be Hausdorff; this follows from Lemma 2.1.

Very briefly, the proof of Theorem 3.3 goes as follows. The bounds on l_μ and l_ν allow one to use Thurston [23] Theorem 3.3 (as in the proof of the double limit theorem) to

conclude the existence of an algebraic limit G_∞ . The main part of the work is to show that G_∞ is quasifuchsian. One uses continuity of the lamination length to show that the laminations μ and ν are realised in the limit 3-manifold \mathbb{H}^3/G_∞ . By carefully using the full force of the algebraic convergence, one can then show that, in the universal cover \mathbb{H}^3 , the lifts of the geodesic laminations $|\mu|$ and $|\nu|$ from $\mathbb{H}^3/G(q_n)$ approach the lifts of the realisations of $|\mu|$ and $|\nu|$ from \mathbb{H}^3/G_∞ in the Hausdorff topology. From this it is not hard to deduce that the laminations $|\mu|$ and $|\nu|$ lie in boundary of the convex core of \mathbb{H}^3/G_∞ . One deduces that this boundary has two components (or one two-sided component if G_∞ is Fuchsian) from which it follows that $G_\infty \in \mathcal{QF}$. Inspection shows that exactly the same proof gives the more general version Theorem 3.4, provided we have Hausdorff convergence of $|\mu_n|$ to $|\mu|$ and $|\nu_n|$ to $|\nu|$.

Alternatively, one can modify the proof of the Lemme de fermeture [3] Proposition 8. They make an assumption on the limit bending laminations (which in particular rules out that the limit group is Fuchsian) but since the main use of this assumption is to get a bound on the lengths l_{β^\pm} , their proof can be adapted relatively easily to our case.

3.0.5 Scaling functions

For fixed $\mu, \nu \in \mathcal{ML}$ we define the *scaling functions* ξ_μ, ξ_ν by $\beta^+ = \xi_\mu \mu$ and $\beta^- = \xi_\nu \nu$.

Proposition 3.5 *The scaling function ξ_μ is real analytic on $\mathcal{P}(\mu, \nu)$.*

PROOF. Suppose first that $\mu \in \mathcal{ML}_\mathbb{Q}$ and let $\gamma = |\mu|$. Let $\lambda_\gamma, \tau_\gamma$ be complex Fenchel Nielsen coordinates for \mathcal{QF} with respect to γ . Here λ_γ is the complex length of γ and τ_γ is the complex twist along γ , see for example [12] or [11]. The quasifuchsian group G can be described up to conjugation by these parameters, which are holomorphic functions on \mathcal{QF} . As explained in detail in [12], if $\beta^+ \in [\mu]$ then $\beta^+ = \theta_\gamma \delta_\gamma$ and $\theta_\gamma = \Im \tau_\gamma$. This easily gives the result.

For general μ , we can obtain any group for which $\beta^+ \in [\mu]$ by a complex earthquake along μ . The scale factor ξ_μ is easily seen to be $\Im \tau_\mu$. This function is again holomorphic, see Section 7.3 of [12]. One can also obtain this result by using the shear-bend coordinates developed by Bonahon in [2], see also [1]. \square

In the special case under discussion the bending measure conjecture can thus be viewed as the assertion that the map $\mathcal{P}(\mu, \nu) \rightarrow \mathbb{R}^+ \times \mathbb{R}^+$, $(l_\mu, l_\nu) \mapsto (\xi_\mu, \xi_\nu)$ is injective.

4 Global geometry

We shall need various results which control the behaviour of bending angles versus bending lengths. The following basic inequality is due to Bridgeman:

Proposition 4.1 ([4] Proposition 2) *There exists a universal constant $K > 0$ such that if $\beta^+ \in [\mu] \in \mathcal{PML}$, then $l_\mu \xi_\mu \leq K$.*

Say $G \in \mathcal{QF}$. If $\gamma \in \pi_1(S)$ we let γ^* be the geodesic representative of γ in \mathbb{H}^3/G and we write γ^\pm for the geodesic representative in the hyperbolic structure of $\partial\mathcal{C}^\pm$. Denote the respective lengths by l_{γ^*} and l_{γ^\pm} . We recall that $l_{\gamma^*} = \Re l_{\gamma^*}$ where l_{γ^*} is the complex length defined in Section 3. The following result, a simplified version of Lemma A.1 of [17], see also Proposition 5.1 in [21], gives a good comparison between the lengths of γ^+ and γ^* when the intersection of γ with the bending measure is reasonably small.

Proposition 4.2 *Let β^+ be the bending measure of the component $\partial\mathcal{C}^+/G$ of the convex hull boundary of the manifold \mathbb{H}^3/G . Let γ be a simple closed curve with geodesic representatives γ^* in \mathbb{H}^3/G and γ^+ in $\partial\mathcal{C}^+$. Then there exist $A, B > 0$ such that if $i(\gamma, \beta^+) < \pi/12$, then $l_{\gamma^+} < Al_{\gamma^*} + B$.*

Now we can prove a result which will give the global control we need.

Proposition 4.3 *Fix $a > 0$. Then for any $K > 0$, there exists $\epsilon > 0$ such that whenever $q \in \mathcal{P}(\mu, \nu)$ and $l_\mu(q) < \epsilon$ with $\xi_\mu(q) = a$, then $l_\nu(q) > K$.*

PROOF. If the result is false, then there is a sequence $q_m \in \mathcal{P}(\mu, \nu)$ for which $\xi_\mu(q_m) = a$ and $l_\nu(q_m) \leq K$ but $l_\mu(q_m) \rightarrow 0$. By Theorem 3.3, under these conditions the groups $G(q_m)$ have an algebraic limit so that in particular, the hyperbolic lengths $l_{\gamma^*}(q_m) = \Re l_{\gamma^*}(q_m)$ are uniformly bounded above for any simple closed curve $\gamma \in \pi_1(S)$.

Suppose first that $\mu \in \mathcal{ML}_\mathbb{Q}$, $\mu = c\delta_\eta$ for $\eta \in \mathcal{S}$. Then the bending angle θ_η is $c\xi_\mu$, where necessarily $\theta_\eta < \pi$. Since S is a punctured torus, we can choose a simple closed curve γ such that $i(\gamma, \eta) = 1$. As on p.195 in [20], for any $G \in \mathcal{QF}$, we have the equation

$$\cosh \tau_\eta/2 = \cosh l_{\gamma^*}/2 \tanh l_{\eta^*}/2$$

where τ_η is the complex Fenchel Nielsen twist along η . Since η is assumed to be the bending locus of $\partial\mathcal{C}^+$, we have $\tau_\eta = t_\eta + i\theta_\eta$ (see [12] or [11] for detailed discussion). Noting that $l_{\eta^*} = l_{\eta^*} \in \mathbb{R}$ and taking real parts we see that if $\theta_\eta = ac < \pi$ is fixed, then $l_{\eta^*} \rightarrow 0$ implies that $l_{\gamma^*} = \Re l_{\gamma^*} \rightarrow \infty$. This contradicts the uniform upper bound to the lengths $l_{\gamma^*}(q_m)$ from the first paragraph.

Now suppose that $\mu \notin \mathcal{ML}_\mathbb{Q}$ and that $\xi_\mu = a$ is fixed. By following a leaf of $|\mu|$ which returns sufficiently close to itself along some transversal, we can choose a simple closed curve γ such that $i(\gamma, \beta^+) = ai(\gamma, \mu) < \pi/12$. Since μ is the bending measure of $\partial\mathcal{C}^+(q_m)$, the lengths l_{μ^*} and l_{μ^+} of the lamination μ in $\mathbb{H}^3/G(q_m)$ and on the pleated surface $\partial\mathcal{C}^+(q_m)$ coincide. Since $l_{\mu^+}(q_m) \rightarrow 0$, by Corollary 2.5 the hyperbolic structures of the pleated surfaces $\partial\mathcal{C}^+(q_m)$ tend to $[\mu] \in \mathcal{PML}$. Since $i(\gamma, \mu) > 0$, it follows that $l_{\gamma^+}(q_m) \rightarrow \infty$. By Proposition 4.2, we have $l_{\gamma^*}(q_m) \rightarrow \infty$, and the same contradiction as before completes the proof. \square

5 Monotonicity of angle for fixed length

Suppose that the bending laminations μ, ν are rational, supported by simple closed curves $\gamma, \delta \in \mathcal{S}$. Theorem 3.1 asserts that the pleating variety $\mathcal{P}(\mu, \nu) = \mathcal{P}(\gamma, \delta)$ is an open real 2-manifold parameterised by the lengths l_γ, l_δ . Theorem 2.6 shows on the other hand that the bending angles $\theta_\gamma, \theta_\delta$ are global coordinates for $\mathcal{P}(\gamma, \delta)$.

In [6] we studied the relationship between bending angle and length for general hyperbolic 3-manifolds with boundary, where the bending laminations were rational. In particular, we showed in Proposition 7.1, that if we regard the lengths l_i of the bending lines as functions of the bending angles θ_i , then the Jacobian matrix $\left(\frac{\partial l_i}{\partial \theta_j}\right)$ evaluated at any point in the relevant pleating variety is negative definite and symmetric. (The main point of [6] is to establish that in general lengths are parameters. Given that both lengths and angles are parameters, so that the Jacobian is non-singular, the fact that it is symmetric and negative definite follows easily from the Schläfli formula for variation of volume of the convex core and using the symmetry of second derivatives, see [6].)

Since the inverse of a negative definite symmetric matrix is also negative definite and symmetric, and since the diagonal entries of a negative definite matrix cannot vanish, we deduce in our special case that $\frac{\partial \theta_\gamma}{\partial l_\gamma}(q) < 0$ for $q \in \mathcal{P}(\gamma, \delta)$ (where the partial derivative is taken keeping l_δ fixed). Recall from Corollary 3.2 that $L_c = l_\nu^{-1}(c) \subset \mathcal{P}(\mu, \nu)$ can be regarded as a line parameterised by l_μ .

Proposition 5.1 *Let $\mu, \nu \in \mathcal{ML}_\mathbb{Q}$. Fix $c > 0$. Then the scaled bending angle ξ_μ is a strictly monotone function of l_μ on the line L_c .*

We want to take limits to prove the same result for general μ, ν . We use the following simple fact about analytic functions:

Lemma 5.2 *Let f_n be a sequence of real valued monotonic functions on $(a, b) \subset \mathbb{R}$ which converges pointwise to a real analytic function $f : (a, b) \rightarrow \mathbb{R}$. The f is either strictly monotonic or constant.*

Notice that a *real analytic* function f may be strictly monotonic even though its derivative vanishes at some points. (The sequence $f_n(x) = x^3 + x/n$ is an instructive example.) Thus in the following result we only claim monotonicity and not necessarily that $\frac{\partial \xi_\mu}{\partial l_\mu} < 0$.

Proposition 5.3 *Let $\mu, \nu \in \mathcal{ML}$. Fix $c > 0$. Let $L_c = l_\nu^{-1}(c) \subset \mathcal{P}(\mu, \nu)$. Then the function ξ_μ is a strictly monotone function of l_μ on the line L_c .*

PROOF. Choose $\mu_n, \nu_n \in \mathcal{ML}_\mathbb{Q}$ with $\mu_n \rightarrow \mu$ and $\nu_n \rightarrow \nu$ in \mathcal{ML} . If either μ or ν is in $\mathcal{ML}_\mathbb{Q}$ then choose the corresponding sequence to be constant. By Lemma 2.1, both sequences of laminations also converge in the Hausdorff topology.

Fix $b, c > 0$ such that $(b, c) \in \mathcal{R}(\mu, \nu)$. By Theorem 3.1, there is a unique point $w(b, c)$ with $l_\mu = b$ and $l_\nu = c$ in $\mathcal{P}(\mu, \nu)$. Since the lines of minima vary continuously with the

laminations, for sufficiently large n , $\mathcal{R}(\mu_n, \nu_n)$ is close to $\mathcal{R}(\mu, \nu)$ and hence there is a unique point $w_n(b, c)$ in $\mathcal{P}(\mu_n, \nu_n)$ for which $l_{\mu_n} = b$ and $l_{\nu_n} = c$. By the strengthened version of the limit pleating theorem 3.4, up to extracting a subsequence these points converge to a point in $\mathcal{P}(\mu, \nu)$. At this point $l_\mu = b$ and $l_\nu = c$, hence by Theorem 3.1 the limit point is unique and must equal $w(b, c)$. Now keep c fixed and vary b . For each n , we know from Proposition 5.1 that $\xi_{\mu_n}(w_n(b, c))$ is a strictly monotone decreasing function of b . By continuity we have $\xi_{\mu_n}(w_n(b, c)) \rightarrow \xi_\mu(w(b, c))$. Hence for c fixed, $\xi_\mu(w(b, c))$ is monotone decreasing. Moreover for fixed $l_\nu = c$, the function ξ_μ is real analytic in l_μ . Hence it is either strictly decreasing or constant. However ξ_μ is certainly not constant since $\xi_\mu \rightarrow 0$ as we approach Fuchsian space along L_c . The result follows. \square

We remark that a somewhat simpler version of the above proof would show monotonicity of angle in the Riley slice of Schottky space, proving the bending measure conjecture for a genus two handlebody with both handles pinched (so that the boundary is a sphere with four punctures).

6 The constant angle variety

Fix $a > 0$ and let $V_a = \{q \in \mathcal{P}(\mu, \nu) \mid \xi_\mu = a\}$. Notice that $V_a \neq \emptyset$ by Theorem 2.6.

Lemma 6.1 V_a contains no isolated points.

PROOF. Suppose the contrary, and let $q \in \mathcal{P}(\mu, \nu)$ be an isolated point of V_a . Then there exists an open disk D containing q with $V_a \cap D = \{q\}$. The image of $D - \{q\}$ under ξ_μ is connected, and it follows that a is a local maximum or minimum for the function ξ_μ restricted to the line $L_c = l_\nu^{-1}(c)$, $c = l_\nu(q)$. This contradicts the strict monotonicity of ξ_μ on L_c from Proposition 5.3. \square

Lemma 6.2 V_a is a real algebraic variety of dimension 1.

PROOF. By definition, V_a has dimension $2 - r$ where r is the maximum rank of $(\frac{\partial \xi_\mu}{\partial l_\mu}, \frac{\partial \xi_\mu}{\partial l_\nu})$ on V_a . Clearly $r \leq 1$. If $r = 0$ then the Jacobian vanishes identically on V_a , from which we deduce that all partial derivatives of ξ_μ vanish identically. This would mean the real analytic function ξ_μ was constant in a neighbourhood of V in $\mathcal{P}(\mu, \nu)$, contradicting Proposition 5.3. \square

We can now use the classical local description of real analytic varieties, see for example Milnor [18] Lemma 3.3:

Theorem 6.3 *Let $P = (x_0, y_0)$ be a non-isolated point in a real one dimensional algebraic variety $V \subset \mathbb{R}^2$. Then there is a neighbourhood of P in which V consists of a finite number*

of branches, each of which is homeomorphic to an open interval on \mathbb{R} . After interchanging the two coordinates if necessary, we may assume this homeomorphism has the form $t \mapsto (x_0 + t^k, y_0 + \sum_{r=1}^{\infty} b_r t^r)$ where the highest common factor of k and the indices of the non-vanishing coefficients b_r is 1.

Proposition 6.4 V_a consists of a unique connected component with no branch points which intersects each line L_c in a unique point.

PROOF. We already know by Proposition 5.3 that $V_a \cap L_c$ consists of at most one point. In other words, the restriction of l_ν to V_a is injective, hence in particular, has no local maxima or minima. Denote this restriction by f . It is easy to see that injectivity of f implies that V_a has no branch points.

Let W be a connected component of V_a . We claim that the image $f(W)$ is $(0, \infty)$. It follows from Theorem 6.3 and the above observation about local maxima and minima that $f(W)$ is open. Now we show that f is proper. Choose $w_n \in W$ with $f(w_n) \rightarrow b$ where $0 < b < \infty$, so that by definition the lengths $l_\nu(w_n)$ are bounded above. Since $\xi_\mu = a$ on V_a , it follows from Bridgeman's inequality that the lengths $l_\mu(w_n)$ are also uniformly bounded above. By Theorem 3.3, the corresponding sequence of groups $G(w_n)$ converge algebraically to a group G and since G is clearly not Fuchsian, either $G \in \mathcal{P}(\mu, \nu)$, or $f(w_n) = l_\nu(w_n) \rightarrow 0$ or $l_\mu(w_n) \rightarrow 0$. By Proposition 4.3, if $l_\mu(w_n) \rightarrow 0$ then $l_\nu(w_n) \rightarrow \infty$, ruling out the last possibility. If $G \in \mathcal{P}(\mu, \nu)$ then by continuity G corresponds to a point $w \in W$ with $f(w) = b$. This shows that f is proper and hence that $f(W) = (0, \infty)$ as claimed.

Thus $W \cap L_c$ is non-empty for each $c \in (0, \infty)$. If V_a had any other connected component, then for some c the intersection $V_a \cap L_c$ would contain more than one point, which is impossible. The result follows. \square

7 Proof of the bending measure conjecture

Suppose $a, b > 0$. Any quasifuchsian group with $\beta^+ = a\mu, \beta^- = b\nu$ necessarily lies in the variety V_a . The following result therefore concludes the proof of the bending measure conjecture:

Proposition 7.1 Fix $a > 0$. The angle ξ_ν is strictly monotonic on V_a .

PROOF. Let $q(c)$ denote the point $V_a \cap L_c$. Choose sequences $\nu_n \rightarrow \nu, \mu_n \rightarrow \mu$, where as usual if $\mu \in \mathcal{ML}_\mathbb{Q}$ we assume that the sequence μ_n is constant. Let $q_n = q_n(c) \in \mathcal{P}(\mu_n, \nu_n)$ be the unique point for which $l_{\nu_n} = c$ and $\xi_{\mu_n} = a$; this exists for large n since $\mathcal{R}(\mu_n, \nu_n)$ is close to $\mathcal{R}(\mu, \nu)$. By Bridgeman's inequality the lengths $l_{\nu_n}(q_n)$ are uniformly bounded above. Hence by the limit pleating theorem, we can extract a subsequence for which $G(q_n)$ converges algebraically to a group G .

We claim that $l_{\mu_n}(q_n)$ does not tend to zero. If $\mu \in \mathcal{ML}_{\mathbb{Q}}$ we can exactly follow exactly the argument in the second paragraph of the proof of Proposition 4.3. If $\mu \notin \mathcal{ML}_{\mathbb{Q}}$, we proceed as in the third paragraph of that proof. Now $[\mu_n]$ is the projective class of the bending measure of $\partial\mathcal{C}^+(q_n)$ so that $l_{\mu_n^+}(q_n)$ and $l_{\mu_n^*}(q_n)$ coincide. By Corollary 2.5, if $l_{\mu_n}(q_n) \rightarrow 0$ then the hyperbolic structures $\partial\mathcal{C}^+(q_n)$ converge to $[\mu]$. This leads to a contradiction to the existence of the algebraic limit of the groups $G(q_n)$ exactly as in Proposition 4.3.

We conclude from Theorem 3.3 that G is represented by a point $q' \in \mathcal{P}(\mu, \nu)$. Moreover since $\xi_{\nu_n}(q_n) \rightarrow \xi_{\nu}(q')$ and $l_{\nu_n}(q_n) \rightarrow l_{\nu}(q')$, we deduce that $q' = q(c)$, so that the limit is independent of the subsequence.

By definition $q_n(c)$ is the point $V_a^n \cap L_c^n$. We already know that the angle function $\xi_{\nu_n}(q_n(c))$ is monotonic in c . We have just shown that $q_n(c) \rightarrow q(c)$ and so $\xi_{\nu_n}(q_n(c)) \rightarrow \xi_{\nu}(q(c))$. We deduce that $\xi_{\nu}(q(c))$ is monotonic in c .

It remains to show that $\xi_{\nu}(q(c))$ is strictly monotonic. Now $\xi_{\nu}(q(c))$ is a real analytic function of the parameters l_{μ}, l_{ν} for $\mathcal{P}(\mu, \nu)$. By Theorem 6.3, on the real analytic variety V_a , each length function is a real analytic function of some $t \in \mathbb{R}$. Thus so is $\xi_{\nu}(q(c))$. We deduce that $\xi_{\nu}(q(c))$ is either strictly monotonic or constant on V_a . To rule out the second possibility, notice that if ξ_{ν} is constant on V_a , then Bridgeman's inequality gives a uniform upper bound to l_{ν} on V_a , contradicting Proposition 6.4. \square

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